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High-Fidelity Linearized J_2 Model for Satellite Formation Flight

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With the recent flurry of research on satellite formation flight, a need has become apparent for a set of linearized equations that describe the relative motion of satellites under the effect of the J_2 geopotential disturbance. In the past, Hill's linearized equations of relative motion have been used to analyze relative motion between satellites, but they were not designed to capture the effect of the J_2 potential. A new set of constant-coefficient, linearized, differential equations of motion is derived. Although surprisingly similar in form to Hill's equations, they are able to capture the effects of the J_2 potential. A numerical simulator is employed to check the fidelity of the equations. It is shown that with the appropriate initial conditions, the new linearized equations of motion have periodic errors of only 0.4% over all inclinations, radii, and cluster configurations. The new linearized equations of motion also allow for insights into the effects of the J_2 disturbance on a satellite cluster including an effect called tumbling, where the cluster as a whole rotates about the vector normal to the orbital plane of the reference orbit. The differential J_2 effects are also analyzed, and the cluster configurations that minimize this effect can be determined from the new equations. Overall, a new high-fidelity set of linearized differential equations is produced that is well suited to model satellite relative motion in the presence of the J_2 potential.

Nomenclature

A	=	amplitude of the cross-track separation
a	=	semimajor axis
B	=	argument of the cross-track separation
c	=	constant defined in Eq. (19)
e	=	eccentricity
f	=	true anomaly
$\boldsymbol{g}$	=	spherical Earth gravitational acceleration
i	=	inclination
J_2	=	second spherical harmonic of Earth's geopotential
$\boldsymbol{J}_2$	=	acceleration due to J_2
k	=	constant defined in Eq. (22)
l	=	linearized time rate of change of A
m	=	initial amplitude of the cross-track motion
n	=	mean motion
q	=	linearized value of B
R_e	=	radius of the Earth
$\boldsymbol{r}$	=	position vector in the inertial coordinate system
s	=	constant defined in Eq. (13)
t	=	time
$\boldsymbol{x}$	=	relative position vector
α, β	=	constant defined in Eq. (42)
γ	=	angular distance between the equator and the intersection of two orbital planes
$\Delta\boldsymbol{x}$	=	relative position between two satellites
$\Delta\Omega$	=	difference in longitude of the ascending nodes
$\delta\gamma$	=	increase in the angular distance between orbital crossings
θ	=	argument of latitude
Φ	=	maximum angular cross-track separation
ϕ	=	initial phasing angle for the cross-track motion
Ω	=	longitude of the ascending node
ω	=	argument of periapsis
$\boldsymbol{\omega}$	=	angular rotation rate of the body-fixed coordinate system

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Introduction

OVER the past few years there has been a growing interest in satellite formation flight, and much research has been done to explore the use of satellite formation flight for such missions as space-based radar or interferometry. With the desire to place spacecraft in formations or "clusters" comes the need to predict accurately and understand the relative motion between the satellites. To describe this relative motion, researchers initially turned to Hill's equations, also know as the Clohessy–Wiltshire equations. Hill's equations are a set of linearized differential equations that describe the relative motion of two spacecraft in similar near-circular orbits (see Ref. 1). This was the logical first step because Hill's equations are very simple to implement and have been successfully used to describe the relative motion of two satellites during rendezvous maneuvers. However, rendezvous maneuvers usually occur over a short time span, and the modeling errors do not have a chance to grow over time. Satellite formation flight missions take place on a much longer timescale. This causes the modeling errors to build up over time and renders the solution useless. Also, because of the simplifying assumptions used when deriving Hill's equations, some of the characteristics of the motion are lost.

The assumption made by Hill's equations that is cited repeatedly as the main source of error is that the Earth is perfectly spherical. Because the Earth is not perfectly spherical, but rather an oblate spheroid (as described by the J_2 potential), modeling errors are introduced by the noncentral forces that result. To remove the modeling errors that are present in Hill's equations, much research has been done with varying success to incorporate the effects of the J_2 potential.

Kechichian[2] derived an exact formulation of the relative motion of satellites in the presence of the J_2 potential by utilizing a reference frame that was subjected to both the J_2 potential and atmospheric drag. However, the resulting equations must be numerically integrated to predict the motion of the satellites over time.

Sedwick et al.[3] incorporated the effects of the J_2 potential as a forcing function that they applied to the right side of Hill's equations. The resulting resonant motion is responsible for the secular drift seen by satellite formations, and the amount of Δv needed to counteract this secular motion is presented.

In papers by Alfriend et al.[4] and by Gim and Alfriend,[5] a state transition matrix is employed to account for the effects of the J_2 potential and slight eccentricities in the reference orbit. This state transition matrix relates the changes in the orbital elements to changes in the local coordinate frame. The papers show that the state transition matrix is an improvement over Hill's equations, but it is also somewhat unwieldy, taking over 10 pages of appendices to represent the matrices.

In a later work by Vadali et al.,[6] an approach in the same vein as that of Sedwick et al.[3] was employed. This approach seeks a linearized combination of Hill's equations with the J_2 effect. The approach by Vadali et al.[6] used a modified reference frame (termed the ghost frame) that tracked the mean drift rates of both a chief and deputy satellite. The resulting equations were linear with periodic coefficients, and differencing them produced a homogeneous set of equations that could be numerically integrated to track the relative motion of two satellites with the same accuracy as their nonlinear simulations for periods up to one day. The magnitude of the error is not explicitly stated.

Previous work by the authors of this paper[7] also partially includes the mean motion of the satellites in defining the reference orbit. The time average of the gradient of the J_2 potential is incorporated to form a new set of constant coefficient linearized equations. However, the time averaging of the gradient of the J_2 potential causes some loss of information in the cross-track direction. Although this error was noted in the paper, a complete solution was not presented.

This paper completes the earlier work by Schweighart and Sedwick.[7] The time average of the gradient of the J_2 potential is once again used to calculate the in-plane relative motion of the satellites. To calculate the cross-track motion, mean variations in the orbital elements and spherical trigonometry are employed. The result of this work is a set of constant-coefficient linear differential equations that can be solved analytically. These equations are

$$\Delta\ddot{x} - 2(nc)\Delta\dot{y} - (5c^2 - 2)n^2\Delta x = 0, \qquad \Delta\ddot{y} + 2(nc)\Delta\dot{x} = 0$$

$$\Delta\ddot{z} + q^2\Delta z = 2lq\cos(qt + \phi) \tag{1}$$

The benefits of these equations are twofold. First, the equations provide a very simple method of modeling the relative motion between satellites. Because they are linear, constant-coefficient differential equations, they can be easily solved to provide analytic solutions. Second, the equations provide insights into the relative motion of satellites under the influence of the J_2 potential that are neither inferable from Hill's equations nor immediately apparent from a simple numerical solution. For example, the period mismatch between the in-plane and cross-track motion produces an effect the authors call tumbling in which the cluster appears to tumble around the orbital angular momentum vector. Also, small differences in the orbital inclination of the satellites in the formation cause differential drift in the longitude of the ascending node. This differential drift causes the satellite formation to break up over time, and the different satellite formations that augment or diminish this effect can be determined from the new equations. All of these topics will be covered in more detail later in the paper. What follows next is a complete derivation of the new equations of motion for satellite formation flight.

Equation Development

The derivation begins with the analytic equation of motion for an orbiting satellite under the influence of the J_2 potential,

$$\ddot{\boldsymbol{r}} = \boldsymbol{g}(\boldsymbol{r}) + \boldsymbol{J}_2(\boldsymbol{r}) \tag{2}$$

where $\boldsymbol{g}(\boldsymbol{r})$ is the gravitational acceleration due to a spherical Earth,

$$\boldsymbol{g}(\boldsymbol{r}) = -(\mu/r^2)\hat{\boldsymbol{r}} \tag{3}$$

$\boldsymbol{J}_2(\boldsymbol{r})$ is the acceleration due to the J_2 potential,[1]

$$\boldsymbol{J}_2(\boldsymbol{r}) = -(3/2)\left(J_2\mu R_e^2/r^4\right)[(1 - 3\sin^2 i\sin^2\theta)\hat{\boldsymbol{x}} + (2\sin^2 i\sin\theta\cos\theta)\hat{\boldsymbol{y}} + (2\sin i\cos i\sin\theta)\hat{\boldsymbol{z}}] \tag{4}$$

$\boldsymbol{r}$ is the position vector of the satellite, and $\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}} - \hat{\boldsymbol{z}}$ is the coordinate system to be described.

A reference orbit is now introduced. For simplicity, a circular reference orbit only under the gravitational influence of a spherical Earth is initially used; however, the following development is not dependent on this assumption. The derivation only requires that the reference orbit is constant radius:

$$\ddot{\boldsymbol{r}}_{\mathrm{ref}} = \boldsymbol{g}(\boldsymbol{r}_{\mathrm{ref}}) \tag{5}$$

A local vertical, local horizontal (LVLH) coordinate system has been assumed with the origin of the coordinate system coinciding with the reference orbit. The $\hat{\boldsymbol{x}}$ vector points in the radial direction; the $\hat{\boldsymbol{z}}$ vector is perpendicular to the orbital plane and points in the direction of the angular momentum vector. Finally, the $\hat{\boldsymbol{y}}$ vector completes the orthogonal triad, and points in the direction of movement. In this $\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}} - \hat{\boldsymbol{z}}$ coordinate system, the stipulation is made that it is a curvilinear coordinate system. The $\hat{\boldsymbol{x}}$ vector remains unchanged; however, the $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{z}}$ vectors curve along a sphere with radius r_{ref}.

Linearizing the gravitational terms in equation (2) with respect to the reference orbit results in

$$\ddot{\boldsymbol{r}} = \boldsymbol{g}(\boldsymbol{r}_{\mathrm{ref}}) + \nabla\boldsymbol{g}(\boldsymbol{r}_{\mathrm{ref}})\cdot\boldsymbol{x} + \boldsymbol{J}_2(\boldsymbol{r}_{\mathrm{ref}}) + \nabla\boldsymbol{J}_2(\boldsymbol{r}_{\mathrm{ref}})\cdot\boldsymbol{x} \tag{6}$$

where the relative position of the satellite taken with respect to the reference orbit is denoted as $\boldsymbol{x}$,

$$\boldsymbol{x} = \boldsymbol{r} - \boldsymbol{r}_{\mathrm{ref}} \tag{7}$$

When a spherical coordinate system $(\hat{\boldsymbol{r}} - \hat{\boldsymbol{\theta}} - \hat{\boldsymbol{i}})$ with the pole aligned with the ascending node is used, the gradient of the $\boldsymbol{g}(\boldsymbol{r})$ gravitational acceleration can be calculated. The result is a second-order tensor:

$$\nabla\boldsymbol{g}(\boldsymbol{r}) = \begin{bmatrix} 2(\mu/r^3) & 0 & 0 \\ 0 & -(\mu/r^3) & 0 \\ 0 & 0 & -(\mu/r^3) \end{bmatrix} \tag{8}$$

The J_2 disturbance [Eq. (4)] is given in $\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}} - \hat{\boldsymbol{z}}$ coordinates. However, the equation can be transformed directly to the $\hat{\boldsymbol{r}} - \hat{\boldsymbol{\theta}} - \hat{\boldsymbol{i}}$ coordinate system without any loss of generality. Taking the gradient results in

$$\nabla\boldsymbol{J}_2(r,\theta,i) = \frac{6\mu J_2 R_E^2}{r_{\mathrm{ref}}^5}\begin{bmatrix} (1 - 3\sin^2 i\sin^2\theta) & \sin^2 i\sin 2\theta & \sin 2i\sin\theta \\ \sin^2 i\sin 2\theta & -\frac{1}{4} - \sin^2 i\left(\frac{1}{2} - \frac{7}{4}\sin^2\theta\right) & -(\sin 2i\cos\theta)/4 \\ \sin 2i\sin\theta & -(\sin 2i\cos\theta)/4 & -\frac{3}{4} + \sin^2 i\left(\frac{1}{2} + \frac{5}{4}\sin^2\theta\right) \end{bmatrix} \tag{9}$$

Because the coordinate system is rotating, rotational terms are needed when calculating the relative acceleration and velocities of the satellites:

$$\ddot{\boldsymbol{x}}_{\mathrm{rel}} = \ddot{\boldsymbol{r}} - \ddot{\boldsymbol{r}}_{\mathrm{ref}} - 2\boldsymbol{\omega}\times\dot{\boldsymbol{x}}_{\mathrm{rel}} - \dot{\boldsymbol{\omega}}\times\boldsymbol{x} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{x}) \tag{10}$$

The rel subscripts will be dropped in the remainder of the text. For a circular reference orbit the rotation rate of the coordinate system is given as

$$\boldsymbol{\omega} = n\hat{\boldsymbol{z}}, \qquad n \equiv \sqrt{\mu/r_{\mathrm{ref}}^3} \tag{11}$$

Substituting Eq. (6) into Eq. (10) and rearranging the terms yields

$$\ddot{\boldsymbol{x}} + 2\boldsymbol{\omega}\times\dot{\boldsymbol{x}} + \dot{\boldsymbol{\omega}}\times\boldsymbol{x} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{r}_{\mathrm{ref}}) + \nabla\boldsymbol{g}(\boldsymbol{r}_{\mathrm{ref}})\cdot\boldsymbol{x} + \boldsymbol{J}_2(\boldsymbol{r}_{\mathrm{ref}}) + \nabla\boldsymbol{J}_2(\boldsymbol{r}_{\mathrm{ref}})\cdot\boldsymbol{x} - \ddot{\boldsymbol{r}}_{\mathrm{ref}} \tag{12}$$

Equation (12) is a linearized differential equation of motion with time-varying coefficients. This is because $\nabla\boldsymbol{J}_2(\boldsymbol{r}_{\mathrm{ref}})$ is not constant except for equatorial orbits. An approximate solution to this problem is to take the time average of the $\nabla\boldsymbol{J}_2(\boldsymbol{r})$ term.

$$\frac{1}{2\pi}\int_0^{2\pi}\nabla\boldsymbol{J}_2(\boldsymbol{r})\,\mathrm{d}\theta = \frac{\mu}{r^3}\begin{bmatrix} 4s & 0 & 0 \\ 0 & -s & 0 \\ 0 & 0 & -3s \end{bmatrix} \tag{13}$$

where

$$s \equiv \left(3J_2R_e^2/8r^2\right)(1 + 3\cos 2i)$$

Equation (12) now becomes

$$\ddot{\boldsymbol{x}} + 2\boldsymbol{\omega} \times \dot{\boldsymbol{x}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{x} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{r}_{\text{ref}}) + \nabla \boldsymbol{g}(\boldsymbol{r}_{\text{ref}}) \cdot \boldsymbol{x} + \boldsymbol{J}_2(\boldsymbol{r}_{\text{ref}}) + \frac{1}{2\pi}\int_0^{2\pi} \nabla \boldsymbol{J}_2(\boldsymbol{r}_{\text{ref}})\, \mathrm{d}\theta \cdot \boldsymbol{x} - \ddot{\boldsymbol{r}}_{\text{ref}} \tag{14}$$

When the time average of the gradient of the J_2 disturbance is taken, the periodic component of the gradient is lost. Although the loss of small periodic motion is not of great concern, there is the possibility that the periodic component of the gradient could resonate with the periodic component of the relative motion. The resulting growth in the solution would also be lost, so that, if it does exist, it must be reestablished.

Growth in the radial direction would signify an ever increasing eccentricity or semimajor axis. According to mean variations in the orbital elements due to the J_2 potential,[1] neither of these two effects exists. Therefore, with regard to radial motion, there is no problem in taking the time average of the gradient of the J_2 potential because there is no secular motion to be lost.

Growth in the tangential direction would signify either an increasing eccentricity (which, as stated earlier, is not possible) or a difference between the orbital period of the satellite and the reference orbit. When the time average of the gradient is taken, some of the information on the orbital period is lost. However, choosing the appropriate initial conditions for the satellite can eliminate this secular motion. If the period of the satellite matches that of the reference orbit, then there cannot be any secular drift in the tangential direction. Partially because of this information loss, the initial velocity conditions will be determined numerically. This is discussed further in a later section.

Finally, growth in the cross-track direction would signify differential changes in the longitude of the ascending node. Although the equations of motion, as derived up to this point, do capture a large component of this motion, some information is still lost. Because of this, the cross-track motion will be derived separately in a later section using the mean variation of the orbital elements.

Overall, although taking the time average of the gradient of the J_2 potential does cause some aspects of the motion to be lost, they will be reaccounted for later in the paper. The equations of motion will also be validated against a numerical simulator to ensure that no secular motion has been overlooked.

Correcting the Orbital Period of the Reference Orbit

For any type of linearization to work, the satellite, which is under the influence of the J_2 potential, must remain in close proximity to the reference orbit. This requires that both the periods and mean radii of the two orbits be similar. The reference orbit, as it is currently defined, is indeed at the mean radius of the satellite, but has the orbital period of an undisturbed satellite not under the influence of the J_2 potential. To keep the satellite and reference orbit in close proximity, we must artificially adjust the period of the referenced orbit to be that of the satellite under the influence of the J_2 potential without changing its radius. Recognizing that the change in orbital period of the disturbed satellite is due to the time averaged effect of the J_2 acceleration, we can evaluate this term and use it to modify the reference orbit directly.

The equation of motion of the reference orbit [Eq. (5)] now becomes

$$\ddot{\boldsymbol{r}}_{\text{ref}} = \boldsymbol{g}(\boldsymbol{r}_{\text{ref}}) + \frac{1}{2\pi}\int_0^{2\pi} \boldsymbol{J}_2(\boldsymbol{r}_{\text{ref}})\, \mathrm{d}\theta \tag{15}$$

where

$$\frac{1}{2\pi}\int_0^{2\pi} \boldsymbol{J}_2(\boldsymbol{r})\, \mathrm{d}\theta = -n^2 r s \hat{\boldsymbol{x}}, \qquad s \equiv \frac{3J_2R_e^2}{8r^2}(1 + 3\cos 2i) \tag{16}$$

Equation (14) then becomes

$$\ddot{\boldsymbol{x}} + 2\boldsymbol{\omega} \times \dot{\boldsymbol{x}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{x} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{x}) = \nabla \boldsymbol{g}(\boldsymbol{r}_{\text{ref}}) \cdot \boldsymbol{x} + \boldsymbol{J}_2(\boldsymbol{r}_{\text{ref}}) + \frac{1}{2\pi}\int_0^{2\pi} \nabla \boldsymbol{J}_2(\boldsymbol{r}_{\text{ref}})\, \mathrm{d}\theta \cdot \boldsymbol{x} - \frac{1}{2\pi}\int_0^{2\pi} \boldsymbol{J}_2(\boldsymbol{r}_{\text{ref}})\, \mathrm{d}\theta \tag{17}$$

Now that the reference orbit has a new period, the angular velocity vector of the rotating coordinate system must also be updated:

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{\text{ref}}) = \boldsymbol{g}(r_{\text{ref}}) + \frac{1}{2\pi}\int_0^{2\pi} \boldsymbol{J}_2(\boldsymbol{r}_{\text{ref}})\, \mathrm{d}\theta \tag{18}$$

which gives the new angular velocity

$$\boldsymbol{\omega} = nc\hat{\boldsymbol{z}}, \qquad c \equiv \sqrt{1+s} \tag{19}$$

Correcting the Reference Orbit for Nodal Drift

Although the preceding equations of motion are a vast improvement over Hill's equations when incorporating the J_2 disturbance, more can be done. Even though the orbital period of the reference orbit has been adjusted to match the period of the satellite under the influence of the J_2 potential, they still drift apart due to separation of the longitude of the ascending node. With reference to Fig. 1, if both the reference orbit and a satellite under the influence of the J_2 potential start at point A, after one orbital period, the satellite will be at point B, whereas the current reference orbit will return to point A. They are now separated by a distance $r_{\text{ref}}\Delta\Omega\sin i$. After two orbital periods, the satellite and the reference orbit will be separated by $r_{\text{ref}}2\Delta\Omega\sin i$, and this process will continue to cause them to drift farther and farther apart. Although the equations of motion in their current form do capture the bulk of this motion, the increasing separation causes the linearization to break down after several periods. Because of this, the reference orbit must again be redesigned so that it tracks this secular motion.

Because the J_2 potential is directly responsible for this separation, the solution to this problem is to determine the aspect of the J_2 potential that causes the drift in the $\hat{z}$ direction and incorporate it into the reference orbit. Using Gauss's variation in the orbital elements (see Ref. 1), it can be shown that the normal component of the J_2 acceleration is solely responsible for the drift in the longitude of the ascending node. Applying the normal component of the J_2 disturbance acceleration to the updated reference orbit [Eq. (15)] results in

$$\ddot{\boldsymbol{r}}_{\text{ref}} = \boldsymbol{g}(\boldsymbol{r}_{\text{ref}}) + \frac{1}{2\pi}\int_0^{2\pi} \boldsymbol{J}_2(\boldsymbol{r}_{\text{ref}})\, \mathrm{d}\theta + [\boldsymbol{J}_2(\boldsymbol{r}_{\text{ref}}) \cdot \hat{z}]\hat{z} \tag{20}$$

where it is understood that the radius of the reference orbit is still to be held constant.

Including this acceleration term makes it difficult to describe analytically the motion of the new reference orbit. The natural choice is to use the mean variation in the orbital elements; however, it will be shown that this is not an adequate approach. The position of the reference point in $\hat{\boldsymbol{X}}$–$\hat{\boldsymbol{Y}}$–$\hat{\boldsymbol{Z}}$ Earth-centered inertial coordinates[1] is

$$\begin{aligned}\boldsymbol{r}_{\text{ref}} = {} & r_{\text{ref}}[\cos\Omega(t)\cos\theta(t) - \sin\Omega(t)\sin\theta(t)\cos i(t)]\hat{\boldsymbol{X}} \\ & + r_{\text{ref}}[\sin\Omega(t)\cos\theta(t) + \cos\Omega(t)\sin\theta(t)\cos i(t)]\hat{\boldsymbol{Y}} \\ & + r_{\text{ref}}[\sin\theta(t)\sin i(t)]\hat{\boldsymbol{Z}}\end{aligned} \tag{21}$$

Approximate expressions for the time variation of the orbital elements of the reference orbit were found by integrating the variations

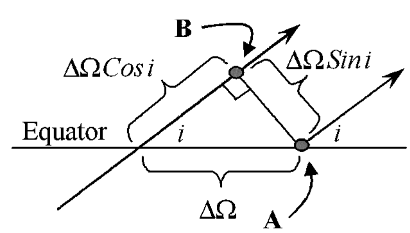

Fig. 1 Effects of a changing longitude of the ascending node.

subject to the normal component of the J_2 acceleration, assuming constant values of the elements over the integration. This results in the following:

$$i(t) = i_{\text{ref}} - \frac{3\sqrt{\mu}J_2R_e^2}{2kr^{\frac{7}{2}}}\cos i_{\text{ref}}\sin i_{\text{ref}}\left[\frac{1-\cos(2kt)}{2}\right]$$

$$\Omega(t) = \Omega_0 - \frac{3\sqrt{\mu}J_2R_e^2}{2kr^{\frac{7}{2}}}\cos i_{\text{ref}}\left[tk - \frac{\sin(2kt)}{2}\right]$$

$$\theta(t) = nct + \frac{3\sqrt{\mu}J_2R_e^2}{2kr^{\frac{7}{2}}}\cos^2 i_{\text{ref}}\left[tk - \frac{\sin(2kt)}{2}\right]$$

$$k = nc + \frac{3\sqrt{\mu}J_2R_e^2}{2r^{\frac{7}{2}}}\cos^2 i_{\text{ref}} \tag{22}$$

When these definitions are inserted into Eq. (21) and simplifications to lowest order in J_2 are performed, it can be shown that an equivalent equation would result if instead the following definitions were used:

$$i(t) = i_{\text{ref}} - \left(3\sqrt{\mu}J_2R_e^2\Big/2kr^{\frac{7}{2}}\right)\cos i_{\text{ref}}\sin i_{\text{ref}}$$

$$\Omega(t) = \Omega_0 - \left(3\sqrt{\mu}J_2R_e^2\Big/2r^{\frac{7}{2}}\right)\cos i_{\text{ref}}t, \qquad \theta(t) = kt \tag{23}$$

Not surprisingly, the expressions for $\Omega(t)$ and $\theta(t)$ are simply what would have resulted from using the mean variations of the elements, as suggested earlier. The interesting result is in the expression for inclination. It can be seen from Eq. (22) that the inclination has a time-varying component over an orbit, and the mean is not actually equal to i_{ref}. One might assume that the mean value would be appropriate for defining the reference orbit. However, the expression in Eq. (23) represents the minimum value. The reason for this has been identified as a geometric coupling between the time variations of the longitude of the ascending node, the argument of latitude, and the inclination. Using the mean inclination results in a much greater error when compared to the actual motion, which is why simply describing the reference orbit in terms of the mean motion is inadequate. Equation (23) can now be substituted back into Eq. (21) to provide an approximate equation of motion for the reference orbit.

Because we have added a component of the J_2 potential to the reference orbit, that component must be subtracted from the equations of motion of the satellite relative to the reference orbit. The resulting equations of motion are

$$\begin{aligned}\ddot{\boldsymbol{x}} + 2\boldsymbol{\omega}\times\dot{\boldsymbol{x}} + \dot{\boldsymbol{\omega}}\times\boldsymbol{x} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{x}) &= \nabla\boldsymbol{g}(\boldsymbol{r}_{\text{ref}})\cdot\boldsymbol{x} + \boldsymbol{J}_2(\boldsymbol{r}_{\text{ref}})\\ &+ \frac{1}{2\pi}\int_0^{2\pi}\nabla\boldsymbol{J}_2(\boldsymbol{r}_{\text{ref}})\,\mathrm{d}\theta\cdot\boldsymbol{x}\\ &- \frac{1}{2\pi}\int_0^{2\pi}\boldsymbol{J}_2(\boldsymbol{r}_{\text{ref}})\,\mathrm{d}\theta - [\boldsymbol{J}_2(\boldsymbol{r}_{\text{ref}})\cdot\hat{z}]\hat{z}\end{aligned} \tag{24}$$

Substituting the appropriate terms into the equations results in an intermediate set of differential equations of the form

$$\begin{aligned}&\ddot{x} - 2(nc)\dot{y} - (5c^2-2)n^2x = -3n^2J_2\left(R_e^2/r_{\text{ref}}\right)\\ &\quad\times\left[\tfrac{1}{2} - 3\sin^2 i_{\text{ref}}\sin^2(kt)/2 - (1+3\cos 2i_{\text{ref}})/8\right]\\ &\ddot{y} + 2(nc)\dot{x} = -3n^2J_2\left(R_e^2/r_{\text{ref}}\right)\sin^2 i_{\text{ref}}\sin(kt)\cos(kt)\\ &\ddot{z} + (3c^2-2)n^2z = 0\end{aligned} \tag{25}$$

Correcting Cross-Track Motion

As stated earlier, the time averaging of the gradient of the J_2 potential causes errors in the cross-track motion. Equation (25) does provide for a fairly accurate model of the cross-track motion, but it has an error in its period and does not capture the secular motion that is normally present in the cross-track direction.

Under the influence of the J_2 potential, the satellite's orbital plane rotates around the $\hat{\boldsymbol{Z}}$ axis (the north pole). This is because the J_2 potential is symmetric across the equator. As a direct result of taking the time average of the gradient of the J_2 potential, the new linearized equations of motion, instead of predicting that the orbital plane rotates around the $\hat{\boldsymbol{Z}}$ axis, predict that the orbital plane rotates around the vector normal to the reference orbit. The time averaging of the gradient of the J_2 potential causes the potential to appear symmetrical about the current reference orbital plane. The result of this modeling error by the equations is that the period of the cross-track motion is incorrect for inclinations other than equatorial.

Overview

Cross-track motion is due solely to that the satellite's orbit and the associated reference orbit are not coplanar. It is a periodic motion that is equal to zero when the two orbital planes intersect and is at a maximum 90 deg away from the intersection of the planes. The location of the intersection of the two orbital planes is based on differences in the inclination and separation in the longitude of the ascending node.

Under the influence of the J_2 potential, these orbital planes rotate about the $\hat{\boldsymbol{Z}}$ axis at different rates. As they move, both the argument and the amplitude of the cross-track motion change. This change is not linear, and spherical trigonometry must be used to derive the cross-track motion. Once the argument $B(t)$ and amplitude $A(t)$ are known as functions of time, the equation for out-of-plane motion can be written in the form

$$z = A(t)\{\sin[B(t)t + \phi]\} \tag{26}$$

where ϕ is the phase angle of the satellite within the cluster.

The following derivation requires the knowledge of the inclination of the satellite and the initial separation of the longitude of the ascending node from that of the reference orbit. When it is assumed that the cluster is initialized at the equator, these parameters can be approximated by using the satellite's initial conditions z_0 and $\dot{z}_0$:

$$i_{\text{sat}} = \dot{z}_0/(kr_{\text{ref}}) + i_{\text{ref}} \tag{27}$$

$$\Delta\Omega_0 = \sin^{-1}\left[\frac{\sin(z_0/r_{\text{ref}})}{\sin i_{\text{ref}}}\right] \approx \frac{z_0}{r_{\text{ref}}\sin i_{\text{ref}}} \tag{28}$$

Argument $B(t)$

The argument of the cross-track motion is dependent on the length of time from when the satellite crosses the intersection of the two orbital planes and returns to that same intersection approximately one orbital period later. When the orbital planes move, the location of this intersection changes, as does the argument of the cross-track term. The angular distance between the intersections of the orbital planes from one pass to the next can be calculated. Figure 2 shows the reference orbit and the orbit of the satellite near the equator. When

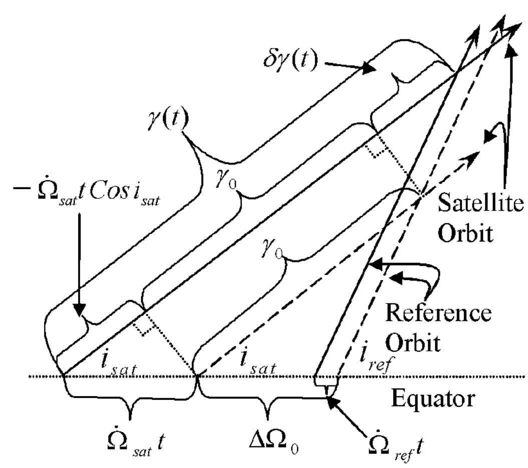

Fig. 2 Moving intersection of the orbital planes.

spherical trigonometry is used, the location of the intersection of the orbital planes, $\gamma(t)$, can be calculated:

$$\gamma(t) = \cot^{-1}\left[\frac{\cot i_{\text{ref}} \sin i_{\text{sat}} - \cos i_{\text{sat}} \cos \Delta\Omega(t)}{\sin \Delta\Omega(t)}\right] \tag{29}$$

where $\Delta\Omega(t)$ is the time-varying separation of the longitude of the ascending nodes. This can be determined by the following:

$$\Delta\Omega(t) = \Omega_{\text{sat}} - \Omega_{\text{ref}} = \Delta\Omega_0 + (\dot{\Omega}_{\text{sat}} - \dot{\Omega}_{\text{ref}})t$$

$$\dot{\Omega}_{\text{sat}} = -\frac{3}{2}\frac{J_2 n R_e^2}{r_{\text{ref}}^2}\cos i_{\text{sat}}, \qquad \dot{\Omega}_{\text{ref}} = -\frac{3}{2}\frac{J_2 n R_e^2}{r_{\text{ref}}^2}\cos i_{\text{ref}} \tag{30}$$

In Fig. 2, the dashed lines represent the initial orbital planes, whereas the solid lines are the orbital planes at a later time. The extra distance a satellite must travel to the next intersection with the reference orbit $\delta\gamma(t)$, due to the rotation of the orbital planes about $\hat{Z}$, can be calculated by

$$\delta\gamma(t) = \gamma(t) + \dot{\Omega}_{\text{sat}} t \cos i_{\text{sat}} - \gamma_0$$

$$\delta\dot{\gamma}(t) = \dot{\gamma}(t) + \dot{\Omega}_{\text{sat}} \cos i_{\text{sat}} \tag{31}$$

Once the parameter $\delta\gamma(t)$ has been calculated, the argument of the cross-track motion can be written as

$$B(t) = nc - \delta\dot{\gamma}(t) \tag{32}$$

Whereas Eq. (31) provides an accurate means of determining $\delta\gamma(t)$, parameter $\gamma(t)$ is a nonlinear function of $\Delta\Omega(t)$ and, thus, nonlinear in time. A first-order approximation can be made by calculating the derivative of $\gamma(t)$ with respect to $\Delta\Omega(t)$. The linearization can be justified because satellites in a cluster will only have small changes in $\Delta\Omega(t)$. A large change in $\Delta\Omega(t)$ would signify that the cluster has separated to a point where it no longer resembles the original cluster:

$$\dot{\gamma}(t) = \frac{\partial\gamma}{\partial\Delta\Omega}\frac{\mathrm{d}\Delta\Omega}{\mathrm{d}t}$$

$$\left(\frac{\partial\gamma}{\partial\Delta\Omega}\right)_{t=0} = \cos\gamma_0 \sin\gamma_0 \cot\Delta\Omega_0 - \sin^2\gamma_0 \cos i_{\text{sat}}$$

$$\frac{\mathrm{d}\Delta\Omega}{\mathrm{d}t} = (\dot{\Omega}_{\text{sat}} - \dot{\Omega}_{\text{ref}}) \tag{33}$$

Combining Eq. (31) and Eq. (33), the parameter $\delta\dot{\gamma}(t)$ is now a constant defined as

$$\delta\dot{\gamma} = \frac{\partial\gamma}{\partial\Delta\Omega}(\dot{\Omega}_{\text{sat}} - \dot{\Omega}_{\text{ref}}) + \dot{\Omega}_{\text{sat}} \cos i_{\text{sat}} \tag{34}$$

Once this has been calculated, the argument of the cross-track motion can be written as

$$B(t) = nc - \delta\dot{\gamma}, \qquad q \equiv nc - \delta\dot{\gamma} \tag{35}$$

Note that due to the preceding linearization, $B(t)$ is a constant (defined as q). Also, if $i_{\text{sat}} = i_{\text{ref}}$, then $nc - \delta\dot{\gamma} = k$.

Equation (33) also allows for some insight into the motion of the location of the orbital plane crossing. When the difference between i_{ref} and i_{sat} is small and $\Delta\Omega_0$ is small, $\partial\gamma/\partial\Delta\Omega$ becomes very large. Thus, small changes in $\Delta\Omega(t)$ can result in large changes in $\delta\gamma$. This motion can be thought of as a scissoring effect. When a pair of scissors is opened, the point of intersection of the two blades moves very rapidly from the tips back. As the handle is opened further, the rate at which the intersection of the two blades moves toward the handle slows down. With orbital planes, the location of the intersection of the orbital planes will move very quickly away from the equator, and then as it approaches the poles, it will slow down.

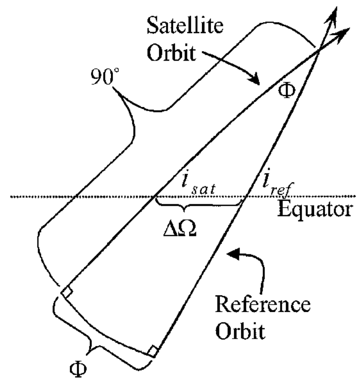

Fig. 3 Determining the amplitude.

Amplitude $A(t)$

The amplitude of the cross-track motion is dependent on the maximum separation between the two different orbital planes. This can once again be calculated with spherical trigonometry. From Fig. 3, it can be seen that the amplitude is based only on the inclination of both orbits, and their separation at the equator.

The angle $\Phi(t)$ can be calculated using

$$\Phi(t) = \cos^{-1}[\cos i_{\text{sat}} \cos i_{\text{ref}} + \sin i_{\text{sat}} \sin i_{\text{ref}} \cos \Delta\Omega(t)] \tag{36}$$

where $\Delta\Omega(t)$ is the time-varying separation of the longitude of the ascending nodes.

Now that $\Phi(t)$ has been determined as a function of time, the amplitude of the out-of-plane motion can be defined as

$$A(t) = r_{\text{ref}}\Phi(t) \tag{37}$$

We can simplify Eq. (37) by linearizing $\Phi(t)$. The linearization is once again justified because $\Phi(t)$ is nonlinear with respect to $\Delta\Omega(t)$ and $\Delta\Omega(t)$ will only undergo small variations.

$$A(t) = lt + m$$

$$l \equiv r_{\text{ref}}\dot{\Phi} = -r_{\text{ref}}\frac{\sin i_{\text{sat}} \sin i_{\text{ref}} \sin\Delta\Omega_0}{\sin\Phi_0}(\dot{\Omega}_{\text{sat}} - \dot{\Omega}_{\text{ref}})$$

$$m \approx r_{\text{ref}}\Phi_0 \tag{38}$$

There is also very useful insight gained by Eq. (38). One would initially assume that secular motion in the cross-track direction (given by l) would be greatest for cases when the difference in inclinations of the satellite and reference orbits are the greatest, providing the most rapid separation of nodal longitude. However, when $\Delta\Omega(t)$ is a maximum, the satellite's orbit intersects the reference orbit at the equator, and, therefore, $\Delta\Omega_0 = 0°$. Equation (38) shows that when $\Delta\Omega_0 = 0°$ then $l = 0$ and that there is no secular motion in the cross-track direction. The initial condition that produces the maximum secular motion in the cross-track direction is when $\phi \approx 45$ deg, where ϕ is the phase angle defined in Eq. (26).

Final Cross-Track Motion

Having derived equations for the amplitude and argument of the cross-track motion, we can now write the out-of-plane motion as

$$z = A(t)\sin[B(t)t + \phi], \qquad z = (lt + m)\sin(qt + \phi) \tag{39}$$

The constants m and ϕ are found using the initial conditions z_0 and $\dot{z}_0$:

$$m \sin\phi = z_0, \qquad l\sin\phi + qm\cos\phi = \dot{z}_0 \tag{40}$$

Final Equations of Motion Relative to the Reference Orbit

Incorporating the corrected cross-track motion into Eq. (25) results in the final linear, constant-coefficient differential equation of motion relating the motion of a satellite to a circular reference orbit.

The equations are initialized so that, at time $t = 0$, the cluster is crossing the equator:

$$\ddot{x} - 2(nc)\dot{y} - (5c^2 - 2)n^2 x = -3n^2 J_2\left(R_e^2/r_{\text{ref}}\right) \times \left\{\tfrac{1}{2} - \left[3\sin^2 i_{\text{ref}} \sin^2(kt)/2\right] - [(1 + 3\cos 2i_{\text{ref}})/8]\right\}$$

$$\ddot{y} + 2(nc)\dot{x} = -3n^2 J_2\left(R_e^2/r_{\text{ref}}\right)\sin^2 i_{\text{ref}} \sin(kt)\cos(kt)$$

$$\ddot{z} + q^2 z = 2lq\cos(qt + \phi) \tag{41}$$

By solving these differential equations, the final equations of motion are presented next. Note that the initial conditions $\dot{x}_0$ and $\dot{y}_0$ have been selected to remove any secular motion or constant offset terms, that is,

$$x = (x_0 - \alpha)\cos\left(nt\sqrt{1-s}\right) + \frac{\sqrt{1-s}}{2\sqrt{1+s}} y_0 \times \sin\left(nt\sqrt{1-s}\right) + \alpha\cos(2kt)$$

$$y = -\frac{2\sqrt{1+s}}{\sqrt{1-s}}(x_0 - \alpha)\sin\left(nt\sqrt{1-s}\right) + y_0 \times \cos\left(nt\sqrt{1-s}\right) + \beta\sin(2kt)$$

$$z = (lt + m)\sin(qt + \phi)$$

with

$$\alpha = -\frac{3J_2 R_e^2 n^2}{4kr_{\text{ref}}}\frac{\left(3k - 2n\sqrt{1+s}\right)}{[n^2(1-s) - 4k^2]}\sin^2 i_{\text{ref}}$$

$$\beta = -\frac{3J_2 R_e^2 n^2}{4kr_{\text{ref}}}\frac{\left[2k\left(2k - 3n\sqrt{1+s}\right) + n^2(3+5s)\right]}{2k[n^2(1-s) - 4k^2]}\sin^2 i_{\text{ref}}$$

$$\dot{y}_0 = -2x_0 n\sqrt{1+s} + \frac{3J_2 R_e^2 n^2}{4kr_{\text{ref}}}\sin^2 i_{\text{ref}}$$

$$\dot{x}_0 = y_0 n\left(\frac{1-s}{2\sqrt{1+s}}\right) \tag{42}$$

Satellite Motion Relative to a Second Satellite

Whereas the motion of a satellite with respect to the circular reference orbit is useful in that it gives the motion of the cluster as a whole, relative position and motion of satellites with respect to each other is the primary need for cluster design. The derivation of the relative motion between two satellites in the cluster will be presented in this section.

Deriving relative motion between two satellites can be accomplished by using the relation

$$\boldsymbol{x}_1 - \boldsymbol{x}_2 = \Delta\boldsymbol{x} \tag{43}$$

where $\boldsymbol{x}_1$ is the relative position of the first satellite with respect to the reference orbit, $\boldsymbol{x}_2$ is the relative position of the second satellite with respect to the reference orbit, and $\Delta\boldsymbol{x}$ is the motion of the first satellite with respect to the second satellite.

Substituting Eq. (24) into Eq. (43) results in the following differential equation of motion:

$$\Delta\ddot{\boldsymbol{x}} + 2\boldsymbol{\omega}\times\Delta\dot{\boldsymbol{x}} + \dot{\boldsymbol{\omega}}\times\boldsymbol{x} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\Delta\boldsymbol{x}) = \nabla\boldsymbol{g}(\boldsymbol{r}_{\text{ref}})\cdot\Delta\boldsymbol{x} + \frac{1}{2\pi}\int_0^{2\pi}\nabla\boldsymbol{J}_2(\boldsymbol{r}_{\text{ref}})\,\mathrm{d}\theta\cdot\Delta\boldsymbol{x} \tag{44}$$

The result is the homogenous form of the in-plane differential equations presented earlier in Eq. (25). The cross-track motion is derived in the same manner as discussed earlier with the only change being the substitution of the inclination and the longitude of the ascending node of the second satellite in place of the reference orbit resulting in new constants q, l, m, and ϕ.

Substituting the appropriate terms and the correction to the cross-track motion into Eq. (44) results in the following differential equations of motion in $\hat{\boldsymbol{x}}$–$\hat{\boldsymbol{y}}$–$\hat{\boldsymbol{z}}$ coordinates:

$$\Delta\ddot{x} - 2(nc)\Delta\dot{y} - (5c^2 - 2)n^2\Delta x = 0, \qquad \Delta\ddot{y} + 2(nc)\Delta\dot{x} = 0$$

$$\Delta\ddot{z} + q^2\Delta z = 2lq\cos(qt + \phi) \tag{45}$$

The equations can be solved to produce the following equations of motion. The initial velocity conditions have been selected in the radial and tangential direction to remove drift and offset terms; thus,

$$\Delta x = \Delta x_0\cos\left(nt\sqrt{1-s}\right) + \frac{\sqrt{1-s}}{2\sqrt{1+s}}\Delta y_0\sin\left(nt\sqrt{1-s}\right)$$

$$\Delta y = -\frac{2\sqrt{1+s}}{\sqrt{1-s}}\Delta x_0\sin\left(nt\sqrt{1-s}\right) + \Delta y_0\cos\left(nt\sqrt{1-s}\right)$$

$$\Delta z = (lt + m)\sin(qt + \phi) \tag{46}$$

with

$$\Delta\dot{x}_0 = \frac{n\Delta y_0}{2}\frac{(1-s)}{\sqrt{1+s}}, \qquad \Delta\dot{y}_0 = -2n\Delta x_0\sqrt{1+s}$$

For clarity, all of the necessary constants needed to use Eqs. (45) and (46) are shown next. The only inputs to these equations are the geopotential constants J_2, R_e, and μ, parameters of the reference orbit r_{ref} and i_{ref}, the inclination of the second satellite i_{sat2}, and the initial conditions Δx_0, Δy_0, Δz_0, and $\Delta\dot{z}_0$. From these inputs, the motion of one satellite with respect to another satellite in the presence of the J_2 potential can be calculated:

$$s = \frac{3J_2 R_e^2}{8r_{\text{ref}}^2}(1 + 3\cos 2i_{\text{ref}})$$

$$c = \sqrt{1+s}$$

$$n = \sqrt{\frac{\mu}{r_{\text{ref}}^3}}$$

$$k = nc + \frac{3nJ_2 R_e^2}{2r_{\text{ref}}^2}\cos^2 i_{\text{ref}}$$

$$i_{\text{sat1}} = \frac{\Delta\dot{z}_0}{kr_{\text{ref}}} + i_{\text{sat2}}$$

$$\Delta\Omega_0 = \frac{\Delta z_0}{r_{\text{ref}}\sin i_{\text{ref}}}$$

$$\gamma_0 = \cot^{-1}\left[\frac{\cot i_{\text{sat2}}\sin i_{\text{sat1}} - \cos i_{\text{sat1}}\cos\Delta\Omega_0}{\sin\Delta\Omega_0}\right]$$

$$\Phi_0 = \cos^{-1}[\cos i_{\text{sat1}}\cos i_{\text{sat2}} + \sin i_{\text{sat1}}\sin i_{\text{sat2}}\cos\Delta\Omega_0]$$

$$\dot{\Omega}_{\text{sat1}} = -\frac{3nJ_2 R_e^2}{2r_{\text{ref}}^2}\cos i_{\text{sat1}}$$

$$\dot{\Omega}_{\text{sat2}} = -\frac{3nJ_2 R_e^2}{2r_{\text{ref}}^2}\cos i_{\text{sat2}}$$

$$q = nc - \left(\cos\gamma_0\sin\gamma_0\cot\Delta\Omega_0 - \sin^2\gamma_0\cos i_{\text{sat1}}\right)(\dot{\Omega}_{\text{sat1}} - \dot{\Omega}_{\text{sat2}}) - \dot{\Omega}_{\text{sat1}}\cos i_{\text{sat1}}$$

$$l = -r_{\text{ref}}\frac{\sin i_{\text{sat1}}\sin i_{\text{sat2}}\sin\Delta\Omega_0}{\sin\Phi_0}(\dot{\Omega}_{\text{sat1}} - \dot{\Omega}_{\text{sat2}})$$

$$m\sin\phi = \Delta z_0$$

$$l\sin\phi + qm\cos\phi = \Delta\dot{z}_0$$

Initial Conditions and Closed-Form Solutions

The solutions to the new linearized equations of motion are dependent on six initial conditions. These initial conditions x_0, y_0, z_0, $\dot{x}_0$, $\dot{y}_0$, and $\dot{z}_0$ are specified as the initial position and velocity of the satellite with respect to the reference orbit or to another satellite in the cluster. Two of the initial conditions, $\dot{x}_0$ and $\dot{y}_0$ were selected to eliminate secular motion and constant offset terms. However, the use of these analytically derived initial conditions does not totally eliminate the drift in the in-plane direction. The reason for the error is threefold, consisting of general linearization, time-averaging of the J_2 potential, and the time-averaging of its gradient. The linearization and time-averaging of the gradient are unavoidable because these are necessary to maintain linear, constant-coefficient equations of motion. It can be argued, however, that a better approximation of the time-average J_2 potential could be used because the average is taken under the assumption of a constant-radius reference orbit.

If the true path of the satellite was used to calculate the time average, then the reference orbit period may in fact be closer to the true period. However, there is no way to unambiguously select the initial conditions of the orbit to be tracked because under the J_2 potential there are no longer circular or otherwise unique orbits. For this reason, it seems most practical to follow the current approach of averaging under the assumption of a constant radius and find a means to better calculate the correct no-drift initial conditions. Although an analytic method using energy conservation was employed with some success for satellites having the same inclination as the reference orbit, a general method for establishing the no-drift condition has not yet been identified. In the examples to follow, the initial conditions are found numerically to isolate errors introduced by the other approximations.

Numerical Results

In the following section, a numerical simulator is employed to verify the validity of the new linearized equations of motion. The numerical simulator takes the initial analytic equations of motion as described in Eq. (2) and integrates the absolute motion of each satellite forward in time. The motion of each satellite is then differenced and converted to an LVLH coordinate system. The results of these numerical simulations are compared to the solution obtained with the new linearized equations of motion.

In the simulations, the satellites are placed into a "free-orbit ellipse." A free-orbit ellipse describes the formation configuration where the projection of the satellites' relative motion in the cross-track direction is a 2 by 1 ellipse, in the in-track direction is a line, and in the radial direction is a circle. This configuration was used by Sedwick et al.,[3] Gim and Alfriend,[5] and others because it projects a constant geometric shape in the radial direction that is useful for many types of missions. This type of configuration will also be used in this paper because it is a good representation of possible relative motion and allows for easy comparison to earlier work.

The results of a numerical simulation with a free-orbit ellipse that projects a 100-m circle in the radial direction at a radius of 7000 km is presented. The initial location of the satellite (the phase angle) was chosen to be 45 deg so that the differential drift in the cross-track direction is at a maximum:

$$r_{\text{ref}} = 7000\ \text{km}, \qquad i_{\text{ref}} = i_{\text{sat2}} = 35\ \text{deg}$$

$$\Delta x_0 = 50\ \text{m}\cos(45\ \text{deg}) = 35.36\ \text{m}$$

$$\Delta y_0 = 100\ \text{m}\sin(45\ \text{deg}) = 70.71\ \text{m}$$

$$\Delta z_0 = 100\ \text{m}\sin(45\ \text{deg}) = 70.71\ \text{m}$$

$$\Delta \dot{z}_0 = (100\ \text{m}/k)\cos(45\ \text{deg}) = 76.32\ \text{mm/s} \tag{47}$$

Figure 4 shows the difference between the results of the numerical simulation and the solutions to the new linearized equations of motion. As we can see from Fig. 4, the modeling error is periodic and does not grow in time. The maximum error is approximately 30 cm in the in-track direction. This is an error of only 0.3% of the 100 m free orbit ellipse. This is a large improvement over Hill's equations, which have periodic errors that grow without bounds.

Instead of proving to the reader that the new equations are valid by showing the individual results from several different configurations, the preceding configuration will be used as a baseline configuration, and each parameter will be varied independently over a range of values. The maximum modeling error over 10 orbital periods will be presented for each parameter varied. By presenting the results of the numerical simulations in this fashion, one can see that the equations hold for all ranges of values as opposed to just a few isolated cases. Also, the error trends for each parameter can be visualized.

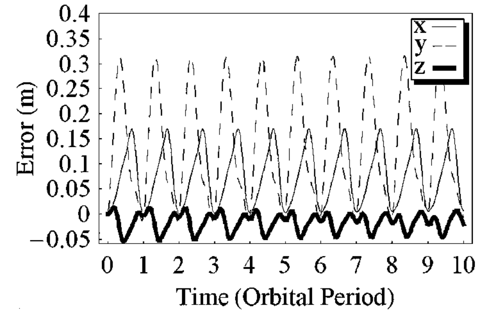

Fig. 4 Results of the numerical simulation.

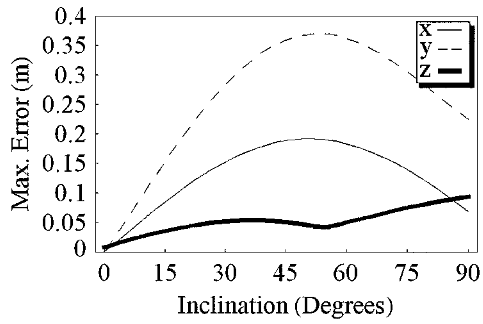

Fig. 5 Maximum modeling error vs inclination of the formation.

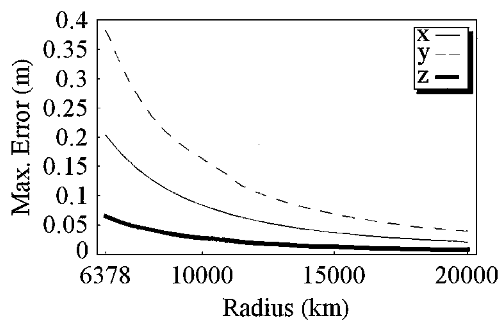

Fig. 6 Maximum modeling error vs radius of the formation.

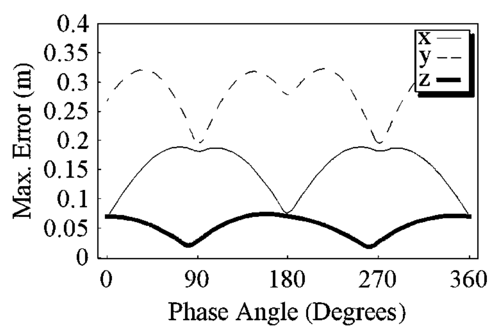

Fig. 7 Maximum modeling error vs phase angle of the satellite.

Figure 5 shows the effect of varying the inclination of the satellites from equatorial to polar orbits. The error is at a minimum for equatorial orbits. This is because the J_2 potential is constant for equatorial orbits and taking the time average of the gradient is not a simplification. With the current baseline configuration, the maximum error is 37 cm in the in-track direction and occurs at inclination near 55 deg.

In Fig. 6, the radius of the satellites' orbit was varied from a height just above the Earth's surface out to a radius of 20,000 km. As can be seen in Fig. 6, the modeling error decreases as the radius of the orbits increases. This is because the J_2 potential is a function of $1/r^4$, whereas the spherical gravitational potential is a function of $1/r^3$, and the effects of the J_2 potential decreases more rapidly as the radius of the orbits increases.

The effect of varying the phase angle is shown in Fig. 7. Once again, there is not a large change in the modeling error, with the maximum error being less than 35 cm.

Although not shown in a figure, the varying of the size of the free-orbit ellipse from 10 to 1000 m has a negligible effect of the modeling error with a maximum value being less than 0.4% of the size of the free-orbit ellipse. Overall, the numerical simulation shows that the new equations of motion are valid for all radii, inclinations, and cluster configurations without much change in error from the baseline configuration.

Tumbling

When a closer look is taken at the new equations of motion, an interesting property can be observed in their solution. Although the solution to the new equations is very similar to that of Hill's equations, one obvious difference is that the period of the cross-track terms is different than the in-plane terms, which are coupled. This difference in period results in a phenomenon coined by the authors as tumbling because the cluster appears to tumble around the $\hat{z}$ axis of the reference orbit. This effect can be better visualized by using a model of satellites in a cluster proposed by Yeh and Sparks.[8] Their model, which uses Hill's equations, states that the motion of satellites in a cluster is on the locus of points described by the intersection of a plane and an elliptical cylinder with the axis of the cylinder along the cross-track direction. There is no restriction on the normal of the plane, but the more interesting solutions result when it is not coincident with the cylindrical axis. The difference in periods between the cross-track terms and the in-plane terms results in the precession of the normal vector of the plane around the cylindrical axis. This motion gives the cluster the appearance of tumbling around the cross-track axis over time. The period of this tumbling is a result of a beating phenomenon between the cross-track and in-plane periods and is much longer than either. For satellite formations that have strict requirements on the projections of the cluster onto the ground, tumbling will cause their projection to degrade over time.

Conclusions

A set of linearized, constant-coefficient differential equations have been developed for describing the relative motion of satellites in the presence of the J_2 potential. They resemble Hill's equations in their form and ease of use, but are capable of accurately and easily describing the relative motion of satellites under the influence of the J_2 potential. These equations were validated using an orbit propagator that incorporates only the J_2 potential and were shown to have only a maximum modeling error of 0.4% over all inclinations, radii, and cluster configurations. The new equations of motion also accurately capture the only type of secular motion that cannot be eliminated with a proper choice of initial conditions: separation in the cross-track direction due to differential nodal effects. These equations can be used to help develop satellite configurations that minimize this effect. Also, the difference in cross-track and in-plane periods results in an effect referred to as tumbling. This motion is an important consideration for clusters that require specific ground projections because the tumbling motion will cause these projections to vary over time. The development of these linear, constant-coefficient equations of motion brings insight to satellite cluster dynamics and provides a tool for developing trajectory optimization and control algorithms.

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